

On the Cayley graph of a commutative ring with respect to its zero-divisors ^{*†}

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Abstract

Let R be a commutative ring with unity and R^+ and $Z^*(R)$ be the additive group and the set of all non-zero zero-divisors of R , respectively. We denote by $\mathbb{CAY}(R)$ the Cayley graph $\text{Cay}(R^+, Z^*(R))$. In this paper, we study $\mathbb{CAY}(R)$. Among other results, it is shown that for every zero-dimensional non-local ring R , $\mathbb{CAY}(R)$ is a connected graph of diameter 2. Moreover, for a finite ring R , we obtain the vertex connectivity and the edge connectivity of $\mathbb{CAY}(R)$. We investigate rings R with perfect $\mathbb{CAY}(R)$ as well. We also study $\text{Reg}(\mathbb{CAY}(R))$ the induced subgraph on the regular elements of R . This graph gives a family of vertex transitive graphs. We show that if R is a Noetherian ring and $\text{Reg}(\mathbb{CAY}(R))$ has no infinite clique, then R is finite. Furthermore, for every finite ring R , the clique number and the chromatic number of $\text{Reg}(\mathbb{CAY}(R))$ are determined.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with unity. Let R be a ring with the additive group R^+ . We denote by $U(R)$, $Z(R)$, $Z^*(R)$, $\text{Reg}(R)$, $\text{Min}(R)$, $\text{Spec}(R)$ and $\text{Max}(R)$, the set of invertible elements, zero-divisors, non-zero zero-divisors, regular elements, minimal prime ideals, prime ideals and maximal ideals of R , respectively. The *Jacobson radical* and the *nilradical* of R are denoted by $J(R)$

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and $\text{Nil}(R)$, respectively. The ring R is said to be *reduced* if it has no non-zero nilpotent element. For a subset X of R , by (X) , we mean *the ideal generated by X* . The *Krull dimension* of R is denoted by $\dim(R)$. By $T(R)$, we mean the *total ring* of R that is the ring of fractions of R with respect to $R \setminus Z(R)$. A *local ring* is a ring with exactly one maximal ideal. A ring with finitely many maximal ideals is called a *semi-local ring*. A ring R is said to be a *von Neumann regular ring*, if for every $x \in R$, there exists $y \in R$ such that $x = x^2y$. The set of *associated prime ideals of R -module R* is denoted by $\text{Ass}(R) = \{\mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} = \text{Ann}(x), \text{ for some } x \in R\}$, where $\text{Ann}(x) = \{y \in R : xy = 0\}$. For classical theorems and notations in commutative algebra, the interested reader is referred to [6] and [20].

Let G be a graph with the vertex set $V(G)$. The *complement* of G is denoted by \overline{G} . For two vertices x and y in a graph G , a walk from x to y is a sequence $xe_1v_1 \cdots v_{l-1}e_ly$, whose terms are alternately vertices and edges of G (not necessarily distinct). We denote this walk by $x - v_1 - \cdots - v_{l-1} - y$. The vertices v_1, \dots, v_{l-1} are called *internal vertices*. We say that two walks (paths) from x to y in a graph G are *vertex internally disjoint* if they share no common internal vertex. If G is connected, then we mean by $\text{diam}(G)$ and $d(x, y)$, the *diameter* of G and the *distance* between two vertices x and y . If G is not connected, then $\text{diam}(G)$ is defined to be ∞ . We denote by K_n the *complete graph* of order n . The *union* of two simple graphs G and H is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a *disjoint union*, and generally denote it by $G + H$. The disjoint union of n copies of G is denoted by nG . We denote the *Cartesian product* of two graphs G and H by $G \square H$. The *direct product* (sometimes called *Kronecker product* or *tensor product*) of two graphs G and H , denoted by $G \times H$, is a graph with the vertex set $V(G) \times V(H)$ and two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if x_1 and x_2 are adjacent in G and y_1 and y_2 are adjacent in H . A *clique* in a graph G is a subset of pairwise adjacent vertices. The supremum of the size of cliques in G , denoted by $\omega(G)$, is called the *clique number* of G . By $\chi(G)$, we denote *the chromatic number* of G i.e. the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A coloring of the vertices such that any two adjacent vertices have different colors is called a *proper vertex coloring*. A graph G is called *k -vertex colorable* if G has a proper vertex coloring with k colors. A graph G is called *perfect* if and only if for every finite induced subgraph H of G , $\chi(H) = \omega(H)$. For $x \in V(G)$ we denote by $N(x)$ the set of all vertices of G adjacent to x . Also, the size of $N(x)$ is denoted by $d(x)$ and is called the *degree* of x . The minimum degree of G is denoted by $\delta(G)$. A graph is called *k -regular*, if all its vertices have degree k . By $N[x]$, we mean $N(x) \cup \{x\}$.

We denote by $\kappa(G)$ and $\kappa'(G)$, the *vertex connectivity* and the *edge connectivity* of G , respectively. For the definitions, see [9]. A graph G is called *vertex transitive* (*edge transitive*) if for every two vertices x and y (two edges e and e') there exists $\rho \in \text{Aut}(G)$ such that $\rho(x) = y$ ($\rho(e) = e'$).

Let G be a group with identity element e and Ω be a non-empty subset of G such that $e \notin \Omega$ and for every $g \in \Omega$, $g^{-1} \in \Omega$. The Cayley graph $\text{Cay}(G, \Omega)$ is a simple graph with the vertex set G and two vertices g and h are adjacent if and only if $g^{-1}h \in \Omega$. A *circulant graph* is a Cayley graph $\text{Cay}(\mathbb{Z}_n^+, \Omega)$, for some $\Omega \subseteq \mathbb{Z}_n \setminus \{0\}$ with property $\Omega = \{-x : x \in \Omega\}$.

Let n be a positive integer, $\mathcal{D} = \{d : 1 \leq d \leq n-1, d \mid n\}$ and T be a subset of \mathcal{D} . The *gcd-graph* $X_n(T)$ has vertices $0, \dots, n-1$ and two vertices x and y are adjacent if and only if $\gcd(x-y, n) \in T$. The concept of gcd-graphs was first introduced by Klotz and Sander, see [17]. In [22], it is shown that integral circulant graphs are exactly the gcd-graphs. For more information on gcd-graphs, we refer the reader to [8], [13], [17] and [22].

The gcd-graph $X_n(\{1\})$ is called *the unitary Cayley graph*, see [10] and [17] and references therein. In [10], the unitary Cayley graph of a commutative ring R is defined as $G_R = \text{Cay}(R^+, U(R))$. For more information on G_R , we refer the reader to [3], [16] and [18]. It is clear that $G_{\mathbb{Z}_n} \cong X_n(\{1\})$.

It is obvious that every gcd-graph $X_n(T)$ with the property $1 \in T$, is of the form $X_n(\{1\}) \cup X_n(T \setminus \{1\})$ and the gcd-graph $X_n(T)$ with $1 \notin T$ is a subgraph of $X_n(\mathcal{D} \setminus \{1\})$, the complement of the unitary Cayley graph $X_n(\{1\})$. Aleksandar Ilić in [12], determined the energy of $X_n(\mathcal{D} \setminus \{1\})$ and proved that this graph is hyperenergetic if and only if n has at least two distinct prime factors and $n \neq 2p$, where p is a prime number. By generalizing the definition of $X_n(\mathcal{D} \setminus \{1\}) \cong \text{Cay}(\mathbb{Z}_n^+, Z^*(\mathbb{Z}_n))$ to a commutative ring R , we study more properties of $X_n(\mathcal{D} \setminus \{1\})$. This generalization can be simply done by $\mathbb{CAY}(R) = \text{Cay}(R^+, Z^*(R))$, a graph whose vertices are elements of R and in which two distinct vertices x and y are joined by an edge if and only if $x - y \in Z(R)$. In Figure 1, $\mathbb{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\mathbb{CAY}(\mathbb{Z}_6)$ are shown.

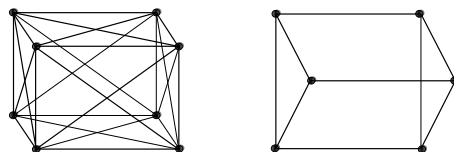


Figure 1. $\mathbb{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\mathbb{CAY}(\mathbb{Z}_6)$

As a graph associated to a commutative ring R , the *total graph* of R , denoted by $T(\Gamma(R))$, is a graph with the vertex set R such that two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. The authors in [4] have studied $T(\Gamma(R))$ and two of its subgraphs $Reg(\Gamma(R))$ and $Z(\Gamma(R))$, the induced subgraphs of $T(\Gamma(R))$ on $Reg(R)$ and $Z(R)$, respectively. However, $T(\Gamma(R))$ and $\mathbb{CAY}(R)$ are close in definition but sometimes they have different properties. For instance, the total graph of a finite ring R can be biregular i.e. with exactly two distinct degrees but we will see that $\mathbb{CAY}(R)$ is always regular. In [21], all finite rings R such that $\mathbb{CAY}(R) \cong T(\Gamma(R))$ are characterized.

For a commutative Noetherian ring R , the chromatic number of $\mathbb{CAY}(R)$, as a simple graph associated with a commutative ring, was independently studied in [7]. Some properties of $\mathbb{CAY}(R)$ such as clique number, independence number, domination number, girth, strongly regularity and edge transitivity have been studied in [1]. In [16], Kiani et al. obtain eigenvalues of $\mathbb{CAY}(R) = \overline{G_R}$ as integers and compute the energy of $\mathbb{CAY}(R)$, for a finite ring R . In [18] among other results, the authors characterize all finite rings R such that $\mathbb{CAY}(R)$ is a Ramanujan graph. In this paper, we continue studying $\mathbb{CAY}(R)$. We also introduce $Reg(\mathbb{CAY}(R))$ the induced subgraph of $\mathbb{CAY}(R)$ on $Reg(R)$. This graph gives a family of vertex transitive graphs. We first present some elementary results on $\mathbb{CAY}(R)$.

Lemma 1. *Let R be a ring. Then the following statements hold:*

- (i) $\mathbb{CAY}(R)$ has no edge if and only if R is an integral domain,
- (ii) If (R, \mathfrak{m}) is an Artinian local ring, then $\mathbb{CAY}(R)$ is a disjoint union of $|\frac{R}{\mathfrak{m}}|$ copies of the complete graph $K_{|\mathfrak{m}|}$,
- (iii) $\mathbb{CAY}(R)$ cannot be a complete graph,
- (iv) $\mathbb{CAY}(R)$ is vertex transitive,
- (v) $\mathbb{CAY}(R)$ is a regular graph of degree $|Z(R)| - 1$ with isomorphic components.

Proof. Parts (i) and (iii) are obvious. Part (ii) follows from $Z(R) = \mathfrak{m}$. Part (iv) holds for every Cayley graph of a group. To prove the last part, note that under an automorphism of graph G , any component of G is isomorphically mapped to another component. Since $\mathbb{CAY}(R)$ is vertex-transitive, we conclude that the components of $\mathbb{CAY}(R)$ are isomorphic and so (v) is proved. \square

Remark 2. $\mathbb{CAY}(R)$ is vertex transitive but it is not necessarily edge transitive. To see this, consider $\mathbb{CAY}(\mathbb{Z}_6) \cong K_2 \square K_3$ which is not edge transitive (see Figure 1).

2. The Connectivity of the Cayley Graph of a Ring

In this section, we study the connectivity of $\mathbb{CAY}(R)$. One of the main results of this section is: for every zero-dimensional non-local ring R , $\mathbb{CAY}(R)$ is a connected graph with diameter 2. To prove this, we first need the following results.

The following lemma has a key role in our proofs. This lemma implies that every element of each minimal prime ideal of a ring is a zero-divisor, see [15, Theorem 84].

Lemma 3. *Let R be a ring. Then the following statements hold:*

- (i) *If $\mathfrak{p} \in \text{Min}(R)$ and $a \in \mathfrak{p}$, then there exists $b \in R \setminus \mathfrak{p}$ such that $ba^i = 0$, for some $i \in \mathbb{N}$. In particular, $\bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p} \subseteq Z(R)$.*
- (ii) *If R is reduced, then $\bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p} = Z(R)$.*

Proof. Let $U = R \setminus \mathfrak{p}$, $V = \{1, a, a^2, \dots\}$ and $S = UV$. Since \mathfrak{p} is a prime ideal, U is a multiplicatively closed subset of R . So, S is a multiplicatively closed subset of R . If $0 \notin S$, then by [20, Theorem 3.44], there exists $\mathfrak{q} \in \text{Spec}(R)$ such that $\mathfrak{q} \cap S = \emptyset$. This yields that $\mathfrak{q} \subseteq R \setminus S \subseteq R \setminus U = \mathfrak{p}$. Thus, $\mathfrak{q} \subseteq \mathfrak{p}$. Since $a \in \mathfrak{p} \cap S$, we conclude that $\mathfrak{q} \subsetneq \mathfrak{p}$. This is a contradiction because $\mathfrak{p} \in \text{Min}(R)$. So, we can assume that $0 \in S$. Thus, there is a natural number i such that $ba^i = 0$, for some $b \in R \setminus \mathfrak{p}$. This implies that $a \in Z(R)$ and the proof of Part (i) is complete. Now, we prove Part (ii). If R is a reduced ring, then by [20, Corollary 3.54], $\bigcap_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p} = \text{Nil}(R) = \{0\}$. Therefore, for every $x \in Z(R)$, there exists $\mathfrak{p} \in \text{Min}(R)$ such that $\text{Ann}(x) \not\subseteq \mathfrak{p}$. Thus, $x \in \mathfrak{p}$ and using Part (i), the proof is complete. \square

Using Part (i) of Lemma 3, we have the following corollary [15, Theorem 91].

Corollary 4. *Let R be a zero-dimensional ring. Then $Z(R)$ is the union of all maximal ideals of R . In particular, $\text{Reg}(R) = U(R)$ and $\mathbb{CAY}(R) = \overline{G_R}$.*

The following lemma is obvious and the proof is omitted.

Lemma 5. *Let R be a ring such that $Z(R)$ is an ideal of R . Then $\mathbb{CAY}(R)$ is a disjoint union of $|\frac{R}{Z(R)}|$ complete graphs $K_{|Z(R)|}$ on the elements of a coset of $Z(R)$.*

So, to argue about the connectedness, we assume that $Z(R)$ is not an ideal.

Theorem 6. *Let R be a ring. Then $\mathbb{CAY}(R)$ is connected if and only if $R = (Z(R))$. Moreover, if $\mathbb{CAY}(R)$ is connected, then the following are equal:*

- (i) $d(0, 1)$,
- (ii) $\text{diam}(\mathbb{CAY}(R))$,
- (iii) The least integer n such that $1 = z_1 + \cdots + z_n$, where $z_i \in Z(R)$, for $1 \leq i \leq n$.

Proof. The proof is very similar to the proof of [4, Theorem 3.4]. Suppose that $\mathbb{CAY}(R)$ is connected. Thus, there exists a path $1 \text{ --- } z_1 \text{ --- } \cdots \text{ --- } z_n \text{ --- } 0$ from 1 to 0. Hence, $1 = (1 - z_1) + (z_1 - z_2) + \cdots + z_n \in (Z(R))$. Conversely, suppose that $1 = a_1 + \cdots + a_m$, where $a_i \in Z^*(R)$. Thus,

$$\sum_{i=1}^m a_i \text{ --- } \sum_{i=1}^{m-1} a_i \text{ --- } \cdots \text{ --- } \sum_{i=1}^2 a_i \text{ --- } a_1 \text{ --- } 0$$

is a walk of length m from 1 to 0. Hence, for every vertex x of R ,

$$x \sum_{i=1}^m a_i \text{ --- } x \sum_{i=1}^{m-1} a_i \text{ --- } \cdots \text{ --- } x \sum_{i=1}^2 a_i \text{ --- } xa_1 \text{ --- } 0$$

is a walk of length m from x to 0. So, there exists a path of length at most m between x and 0. Thus, $\mathbb{CAY}(R)$ is connected. Moreover, $d(0, x) \leq d(0, 1)$, for every vertex x . This argument also shows that $d(0, 1)$ is equal to the least integer n such that 1 is a sum of n elements of $Z(R)$. On the other hand, by the vertex transitivity of $\mathbb{CAY}(R)$, we conclude that for every two distinct vertices u and v , there exists $\varphi \in \text{Aut}(\mathbb{CAY}(R))$ such that $d(u, v) = d(0, \varphi(v)) \leq d(0, 1)$. Hence, $\text{diam}(\mathbb{CAY}(R)) = d(0, 1)$ and the proof is complete. \square

Example 7. By the above criterion and [4, Theorem 3.4], we deduce that if $R = (Z(R))$, then $\text{diam}(\mathbb{CAY}(R)) = \text{diam}(T(\Gamma(R)))$. For every integer $n \geq 2$, [4, Example 3.8], provides a ring R_n whose total graph has diameter n . Hence, for every integer $n \geq 2$, $\text{diam}(\mathbb{CAY}(R_n)) = n$.

The next lemma has a key role in the proof of the main results of this section.

Lemma 8. Let R be a ring and $x \in R$.

- (i) If $x + \text{Nil}(R) \in Z(\frac{R}{\text{Nil}(R)})$, then $x \in Z(R)$. In particular, $\text{diam}(\mathbb{CAY}(R)) \leq \text{diam}(\mathbb{CAY}(\frac{R}{\text{Nil}(R)}))$.
- (ii) Let $\dim(R) = 0$ or R be a Noetherian ring with $\text{Min}(R) = \text{Ass}(R)$. If $x \in Z(R)$, then $x + \text{Nil}(R) \in Z(\frac{R}{\text{Nil}(R)})$. In particular, $\text{diam}(\mathbb{CAY}(\frac{R}{\text{Nil}(R)})) = \text{diam}(\mathbb{CAY}(R))$.

Proof. First suppose that $x + Nil(R) \in Z(\frac{R}{Nil(R)})$. Thus, there exists $y \in R \setminus Nil(R)$ such that $xy \in Nil(R)$. So, there exists a positive integer n such that $(xy)^n = 0$. Since $y \in R \setminus Nil(R)$, there exists a non-negative integer l , $l < n$, such that $x^{n-l}y^n = 0$ and $x^{n-l-1}y^n \neq 0$. This implies that $x \in Z(R)$.

To complete the proof of Part (i), note that if

$$1 + Nil(R) \text{ --- } z_1 + Nil(R) \text{ --- } \cdots \text{ --- } z_n + Nil(R) \text{ --- } Nil(R),$$

is a path from $1 + Nil(R)$ to $Nil(R)$ in $\mathbb{CAY}(\frac{R}{Nil(R)})$, then $1 \text{ --- } z_1 \text{ --- } \cdots \text{ --- } z_n \text{ --- } 0$ is a path from 1 to 0 in $\mathbb{CAY}(R)$. Now, by the argument in the proof of Theorem 6, $diam(\mathbb{CAY}(R)) \leq diam(\mathbb{CAY}(\frac{R}{Nil(R)}))$.

We now prove the Part (ii). First suppose that R is a Noetherian ring with $\text{Min}(R) = \text{Ass}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. By [20, Corollary 9.36], $Z(R) = \cup_{i=1}^n \mathfrak{p}_i$. Since $\frac{R}{Nil(R)}$ is a reduced ring, by Lemma 3, we conclude that $Z(\frac{R}{Nil(R)}) = \cup_{i=1}^n \frac{\mathfrak{p}_i}{Nil(R)}$. Therefore, for every $x \in Z(R)$, $x + Nil(R) \in Z(\frac{R}{Nil(R)})$. Now, suppose that R is a ring with $\dim(R) = 0$. Let $x \in Z(R)$. Thus, by Corollary 4, there exists $\mathfrak{p} \in \text{Min}(R)$ such that $x \in \mathfrak{p}$. Hence, by Lemma 3, there exists $y \in R \setminus \mathfrak{p}$ such that $xy \in Nil(R)$. This implies that $x + Nil(R) \in Z(\frac{R}{Nil(R)})$. Now, similar to the last part of the proof of Part (i), the proof of Part (ii) is complete. \square

Remark 9. Let R be a Noetherian ring R with $\text{Min}(R) \neq \text{Ass}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Since $\text{Min}(R) \subseteq \text{Ass}(R)$, one may assume that $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, for some k , $1 \leq k < n$. By the Prime Avoidance Theorem ([20, Theorem 3.61]), let $x \in \mathfrak{p}_{k+1} \setminus \cup_{i=1}^k \mathfrak{p}_i$. We have the condition $x \in \cup_{i=1}^n \mathfrak{p}_i = Z(R)$ but $x + Nil(R) \notin \cup_{i=1}^k \frac{\mathfrak{p}_i}{Nil(R)} = Z(\frac{R}{Nil(R)})$. Hence, in Part (ii) of the previous lemma, $\text{Min}(R) = \text{Ass}(R)$ is not superfluous.

Note that by Lemma 5 and Corollary 4, if R is a zero-dimensional local ring, then $\mathbb{CAY}(R)$ is a disjoint union of complete graphs. Now, we are in a position to prove the following result.

Theorem 10. *Let R be a zero-dimensional non-local ring. Then $\mathbb{CAY}(R)$ is connected and $diam(\mathbb{CAY}(R)) = 2$.*

Proof. By Lemma 8, one may assume that R is a zero-dimensional reduced ring. Thus, by [15, Exercise 22, p.64], R is a von Neumann regular ring. Since $|\text{Min}(R)| \geq 2$, R has a non-zero zero-divisor, say x . Since R is a von Neumann regular ring, there exists $y \in R$ such that $x = x^2y$ and so $e = xy$ is a non-zero idempotent of R . Note that xe and $1 - e$ are non-zero zero-divisors of R . Since $u = xe + 1 - e$ is a unit of R

with inverse $ye + 1 - e$, we conclude that 1 is a sum of two zero-divisors of R . Thus, by Theorem 6, $\text{diam}(\mathbb{CAY}(R)) \leq 2$. As $\mathbb{CAY}(R)$ is not a complete graph, we have $\text{diam}(\mathbb{CAY}(R)) = 2$. The assertion is proved. \square

The following result is a generalization of [4, Theorem 3.4]. The proof simply follows from Theorem 6, Example 7 and the previous result.

Corollary 11. *Let R be a zero-dimensional non-local ring. Then $T(\Gamma(R))$ is connected and $\text{diam}(T(\Gamma(R))) = 2$.*

Till now, we have studied the connectedness of $\mathbb{CAY}(R)$ for every zero-dimensional ring R . The following example shows that Theorem 10 does not hold for Noetherian rings with at least two minimal prime ideals. Moreover, it shows that the assumptions in the previous theorem are necessary.

Example 12. Let K be a field and $R = \frac{K[x,y]}{(xy)}$. Clearly, R is a Noetherian reduced ring with $\dim(R) = 1$, $\text{Min}(R) = \{(x + (xy)), (y + (xy))\}$ and by Lemma 3, $Z(R) = (x + (xy)) \cup (y + (xy))$. Thus, $\mathbb{CAY}(R)$ is a disjoint union of $|K|$ connected graphs of diameter 2 with the vertex sets $\Gamma_\alpha = \{\alpha + f(x) + g(y) + (xy) : f \in K[x], g \in K[y] \text{ and } f(0) = g(0) = 0\}$, where $\alpha \in K$. Note that for every $\alpha \in K$, the ideals $\frac{(x-\alpha, y)}{(xy)}$ and $\frac{(x, y-\alpha)}{(xy)}$ are distinct maximal ideals of R . So, R is a non-local ring.

In the sequel, for a finite ring R , we obtain the vertex connectivity and edge connectivity of $\mathbb{CAY}(R)$. We first need the following lemmas.

Lemma 13. *Let R be a ring, $x \in R$ and $a \in \text{Nil}(R)$. Then $x + a \in Z(R)$ if and only if $x \in Z(R)$.*

Proof. If $a = 0$, then the assertion is obvious. Thus, assume that $a \neq 0$. First suppose that there exists $0 \neq y \in R$ such that $xy = 0$. Since a is nilpotent, there exists a positive integer n such that $a^n = 0$ and $a^{n-1} \neq 0$. Let $k, k \leq n$, be the biggest positive integer such that $ya^{n-k} \neq 0$ and $ya^{n-k+1} = 0$. Clearly, $ya^{n-k}(x + a) = 0$ and so $x + a \in Z(R)$. Conversely, suppose that $x + a \in Z(R)$. Thus, $x = x + a + (-a) \in Z(R)$, as desired. \square

Let G be a connected graph. A non-empty subset S of vertices of G is called a *vertex cut* of G if $G - S$ (the removal of vertices of S from G) is not connected or has exactly one vertex. Note that by Menger's Theorem, for a finite connected graph G , $\kappa(G)$ is equal to the minimum size of vertex cuts of G (see [25, Theorem 4.2.21]).

Lemma 14. *Let $n \geq 2$ be a positive integer and F_1, \dots, F_n be finite fields. Then*

$$\kappa(\mathbb{CA}\mathbb{Y}(F_1 \times \dots \times F_n)) \geq |Z(F_1 \times \dots \times F_n)| - 1.$$

Proof. We show that for every two distinct vertices X and Y of $\mathbb{CA}\mathbb{Y}(F_1 \times \dots \times F_n)$, there are at least $|Z(F_1 \times \dots \times F_n)| - 1$ pairwise vertex internally disjoint paths (or simply pairwise internally disjoint paths) from X to Y . We prove this by induction on n . By [23], $\kappa(K_{|F_1|} \square K_{|F_2|}) = |F_1| + |F_2| - 2$. Note that $|Z(F_1 \times F_2)| - 1 = |(F_1 \times F_2) \setminus (F_1^* \times F_2^*)| - 1 = |F_1| + |F_2| - 2$, where $F_i^* = F_i \setminus \{0\}$. Since $\mathbb{CA}\mathbb{Y}(F_1 \times F_2) \cong K_{|F_1|} \square K_{|F_2|}$, for $n = 2$ the assertion holds. Now, let the assertion hold for n and let $X = (x_1, \dots, x_{n+1})$, $Y = (y_1, \dots, y_{n+1}) \in F_1 \times \dots \times F_{n+1}$. First we need some notations. We recursively express X and Y by $X = (x_1, \hat{X})$ and $Y = (y_1, \hat{Y})$, where $\hat{X} = (x_2, \dots, x_{n+1}) \in F_2 \times \dots \times F_{n+1}$ and $\hat{Y} = (y_2, \dots, y_{n+1}) \in F_2 \times \dots \times F_{n+1}$, respectively.

If $\hat{X} \neq \hat{Y}$, by induction hypothesis, there exist $|Z(F_2 \times \dots \times F_{n+1})| - 1$ pairwise internally disjoint paths from \hat{X} to \hat{Y} in $\mathbb{CA}\mathbb{Y}(F_2 \times \dots \times F_{n+1})$. It means that for every vertex U adjacent to \hat{X} in $\mathbb{CA}\mathbb{Y}(F_2 \times \dots \times F_{n+1})$, there exists a unique path $P(U)$ (among those $|Z(F_2 \times \dots \times F_{n+1})| - 1$ paths) which contains U . Let $P_t(U)$ be that terminal vertex of $P(U)$ which is adjacent to \hat{Y} . So, $P(U)$ has the following form.

$$P(U) : \hat{X} - U - \dots - P_t(U) - \hat{Y}.$$

Note that $U \neq \hat{X}$ and $P_t(U) \neq \hat{Y}$. If $U = P_t(U)$, we simply mean $\hat{X} - U - \hat{Y}$.

Now, we have the required notations to show that there exist at least $|Z(F_1 \times \dots \times F_{n+1})| - 1$ pairwise internally disjoint paths from X to Y . We need to consider the two following cases.

Case 1. X and Y have at least an equal component, say the first component.

Let $X = (a, \hat{X})$, $Y = (a, \hat{Y})$. Note that in this case $\hat{X} \neq \hat{Y}$. Thus, $X - (a, A) - Y$ is a path from X to Y , for every $A \in F_2 \times \dots \times F_{n+1} \setminus \{\hat{X}, \hat{Y}\}$. Every two paths of this form are internally disjoint. Since X and Y are adjacent, we find $|F_2 \times \dots \times F_{n+1}| - 1$ pairwise internally disjoint paths from X to Y of the following types.

Type 1.1. The single path $X - Y$ of length 1.

Type 1.2. The paths of the form $X - (a, A) - Y$ of length 2, where $A \in F_2 \times \dots \times F_{n+1} \setminus \{\hat{X}, \hat{Y}\}$.

Since $\hat{X} \neq \hat{Y}$, we are allowed to use notation $P_t(U)$ for every vertex U adjacent to \hat{X} . Now, let $b \in F_1 \setminus \{a\}$. We consider the following type of paths from X to Y :

Type 1.3. The paths of the form $X \text{---}(b, U) \text{---}(b, P_t(U)) \text{---} Y$, where $b \in F_1 \setminus \{a\}$ and U is a vertex adjacent to \widehat{X} in $\mathbb{CA}\mathbb{Y}(F_2 \times \cdots \times F_{n+1})$.

If $U = P_t(U)$, we simply mean $X \text{---}(b, U) \text{---} Y$. Since for every two distinct vertices U and V which both are adjacent to \widehat{X} , we have $\{U, P_t(U)\} \cap \{V, P_t(V)\} = \emptyset$, we conclude that all the paths of this type are pairwise internally disjoint. The number of paths of type 3 is $(|F_1| - 1)(|Z(F_2 \times \cdots \times F_{n+1})| - 1)$.

Finally, we consider the next type of the paths.

Type 1.4. The paths of the form $X \text{---}(b, \widehat{X}) \text{---}(b, \widehat{Y}) \text{---} Y$ of length 4, where $b \in F_1 \setminus \{a\}$.

It is clear that the paths of this type form $|F_1| - 1$ pairwise internally disjoint paths from X to Y .

Note that by the construction we give, the paths of the same type are pairwise internally disjoint. Moreover, two paths of different types are internally disjoint. Thus, from these 4 types of paths we obtain

$$(|F_1| - 1)|Z(F_2 \times \cdots \times F_{n+1})| + |F_2 \times \cdots \times F_{n+1}| - 1$$

pairwise internally disjoint paths from X to Y . Since

$$Z(F_1 \times \cdots \times F_{n+1}) = \left(\bigcup_{x \in F_1 \setminus \{0\}} \{x\} \times Z(F_2 \times \cdots \times F_{n+1}) \right) \bigcup \{0\} \times F_2 \times \cdots \times F_{n+1},$$

we deduce that

$$(|F_1| - 1)|Z(F_2 \times \cdots \times F_{n+1})| + |F_2 \times \cdots \times F_{n+1}| - 1 = |Z(F_1 \times \cdots \times F_{n+1})| - 1.$$

Hence, we have $|Z(F_1 \times \cdots \times F_{n+1})| - 1$ pairwise internally disjoint paths from X to Y in this case.

Case 2. For $i = 1, \dots, n + 1$, $x_i \neq y_i$.

We show that we can assume that every $F_i \cong \mathbb{Z}_2$. Suppose that one of the F_i , say F_1 , has at least 3 elements. Similar to the previous case, we consider the following types of paths from X to Y .

Type 2.1. The paths of the form $X \text{---}(b, U) \text{---}(b, P_t(U)) \text{---} Y$, where $b \in F_1 \setminus \{x_1, y_1\}$ and U is a vertex adjacent to \widehat{X} in $\mathbb{CA}\mathbb{Y}(F_2 \times \cdots \times F_{n+1})$.

Type 2.2. The paths of the form $X \text{---}(b, \widehat{X}) \text{---}(b, \widehat{Y}) \text{---} Y$ of length 4, where $b \in F_1 \setminus \{x_1, y_1\}$.

These two types of paths give

$$(|F_1| - 2)(|Z(F_2 \times \cdots \times F_{n+1})| - 1) + |F_1| - 2 = (|F_1| - 2)|Z(F_2 \times \cdots \times F_{n+1})|$$

pairwise internally disjoint paths from X to Y with the internal vertices in $(F_1 \setminus \{x_1, y_1\}) \times F_2 \times \cdots \times F_{n+1}$.

Hence, every possible path from X to Y with vertices in $\{x_1, y_1\} \times F_2 \times \cdots \times F_{n+1}$ is internally disjoint from the paths in Types 2.1 and 2.2. Thus, if we find $|Z(F_1 \times \cdots \times F_{n+1})| - 1 - (|F_1| - 2)|Z(F_2 \times \cdots \times F_{n+1})|$ pairwise internally disjoint paths from X to Y with vertices in $\{x_1, y_1\} \times F_2 \times \cdots \times F_{n+1}$, then we are done. To see this, we first consider the following claim.

Claim. $|Z(F_1 \times \cdots \times F_{n+1})| - 1 - (|F_1| - 2)|Z(F_2 \times \cdots \times F_{n+1})|$ is the number of neighbors of X in $\{x_1, y_1\} \times F_2 \times \cdots \times F_{n+1}$.

To prove the claim, let N_1 be the set of neighbors of X in $(F_1 \setminus \{x_1, y_1\}) \times F_2 \times \cdots \times F_{n+1}$ and N_2 be the set of neighbors of X in $\{x_1, y_1\} \times F_2 \times \cdots \times F_{n+1}$. Clearly $|Z(F_1 \times \cdots \times F_{n+1})| - 1 = |N_1| + |N_2|$ and $|N_1| = (|F_1| - 2)|Z(F_2 \times \cdots \times F_{n+1})|$. This proves the claim.

Hence, to complete the proof, we should find $|N_2|$ pairwise internally disjoint paths from X to Y whose vertices are in $\{x_1, y_1\} \times F_2 \times \cdots \times F_{n+1}$. It is not hard to check that $|N_2|$ is degree of $(1, \hat{X})$ in $\mathbb{CAY}(\mathbb{Z}_2 \times F_2 \times \cdots \times F_{n+1})$. Hence, it suffices to find at least $|Z(\mathbb{Z}_2 \times F_2 \times \cdots \times F_{n+1})| - 1$ pairwise internally disjoint paths from $(1, \hat{X})$ to $(0, \hat{Y})$ in $\mathbb{CAY}(\mathbb{Z}_2 \times F_2 \times \cdots \times F_{n+1})$. Thus, we may assume that $|F_1| = 2$ and by continuing this procedure we can suppose that $F_i \cong \mathbb{Z}_2$, for $i = 1, \dots, n+1$ and $X = (1, \dots, 1)$, $Y = (0, \dots, 0)$. Since for every $z \in Z^*(\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)$, $(1, \dots, 1) \text{ --- } z \text{ --- } (0, \dots, 0)$ is a path from $(1, \dots, 1)$ to $(0, \dots, 0)$, we obtain $|Z(\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)| - 1$ pairwise internally disjoint paths from $(1, \dots, 1)$ to $(0, \dots, 0)$. Hence, by the above argument we can find $|Z(F_1 \times \cdots \times F_{n+1})| - 1$ pairwise internally disjoint paths from X to Y .

This implies that the assertion holds for $n+1$ in both Cases 1 and 2. So by induction the proof is complete. \square

Theorem 15. ([9, Theorem 9.14]) *Let G be a simple connected vertex transitive graph of positive degree d . Then $\kappa'(G) = d$.*

Now, we are in a position to obtain the vertex connectivity and the edge connectivity of $\mathbb{CAY}(R)$. A well-known theorem due to Watkins (see [24, Corollary 1.A]) states that the vertex connectivity of every connected edge transitive graph G equals to $\delta(G)$. In [3], the authors apply this theorem to obtain the vertex connectivity of the unitary

Cayley graph G_R . Unfortunately, $\mathbb{CAY}(R)$ is not necessarily edge transitive. In [1], all finite rings R whose $\mathbb{CAY}(R)$ is edge transitive are characterized. The following result states that $\mathbb{CAY}(R)$ is a reliable network i.e. the vertex connectivity of $\mathbb{CAY}(R)$ equals to degree of regularity, for every finite reduced ring R . Since for every finite graph G , we have $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ (see [25, Theorem 4.1.9] or [9, Exercise 9.3.2]), we deduce that $\mathbb{CAY}(R)$ gives a class of vertex transitive graphs with optimal connectivity.

Theorem 16. *Let R be a finite non-local ring. Then $\kappa(\mathbb{CAY}(R)) = |Z(R)| - |Nil(R)|$ and $\kappa'(\mathbb{CAY}(R)) = |Z(R)| - 1$.*

Proof. Since R is non-local, by Theorem 10, $\mathbb{CAY}(R)$ is connected. Thus, by Theorem 15 we have $\kappa'(\mathbb{CAY}(R)) = |Z(R)| - 1$. Now, we prove that $\kappa(\mathbb{CAY}(R)) = |Z(R)| - |Nil(R)|$.

First, we show that $\kappa(\mathbb{CAY}(R)) \geq |Z(R)| - |Nil(R)|$. To see this, we prove that for every two distinct vertices x and y of $\mathbb{CAY}(R)$, there are at least $|Z(R)| - |Nil(R)|$ pairwise (vertex) internally disjoint paths from x to y . First suppose that $x - y \in Nil(R)$. Thus, by Lemma 13, for every $z \in Z^*(R) \setminus \{x - y\}$, $x - z + y - y$ is a path of length 2 from x to y . Since x and y are adjacent, we deduce that there exist at least $|Z(R)| - 1 \geq |Z(R)| - |Nil(R)|$ pairwise internally disjoint paths from x to y . Now, assume that $x - y \notin Nil(R)$. Clearly, $\frac{R}{Nil(R)}$ is non-local and by [6, Theorem 8.7] (or Chinese Remainder Theorem [6, Proposition 1.10]), $\frac{R}{Nil(R)}$ is a finite product of finite fields. Hence, by Lemma 14, there exist $|Z(\frac{R}{Nil(R)})| - 1$ pairwise internally disjoint paths from $x + Nil(R)$ to $y + Nil(R)$ in $\mathbb{CAY}(\frac{R}{Nil(R)})$. This number is exactly the number of neighbors of $x + Nil(R)$ in $\mathbb{CAY}(\frac{R}{Nil(R)})$. For every neighbor of $x + Nil(R)$ say $x + z + Nil(R)$ let $P(z)$ be the unique path which contains $x + z + Nil(R)$, where $z + Nil(R) \in Z^*(\frac{R}{Nil(R)})$. Assume that $P(z)$ has the following vertices.

$$x + Nil(R) - x_1^{(z)} + Nil(R) - \cdots - x_{k(z)}^{(z)} + Nil(R) - y + Nil(R),$$

where $x_1^{(z)} + Nil(R) = x + z + Nil(R)$. Then by Lemma 8, for every $m \in Nil(R)$,

$$x - x_1^{(z)} + m - \cdots - x_{k(z)}^{(z)} + m - y$$

forms a path from x to y in $\mathbb{CAY}(R)$. We denote this path by $P(z) + m$. As z ranges over $Z^*(\frac{R}{Nil(R)})$ and m ranges over $Nil(R)$, the paths $P(z) + m$ generate pairwise internally disjoint paths from x to y in $\mathbb{CAY}(R)$. Thus, we obtain at least $|Nil(R)|(|Z(\frac{R}{Nil(R)})| - 1)$ pairwise internally disjoint paths from x to y in $\mathbb{CAY}(R)$. By Lemma 8, we have $|Nil(R)|(|Z(\frac{R}{Nil(R)})| - 1) = |Z(R)| - |Nil(R)|$. Hence, we obtain

at least $|Z(R)| - |Nil(R)|$ pairwise internally disjoint paths from x to y in $\mathbb{CAY}(R)$. Therefore, $\kappa(\mathbb{CAY}(R)) \geq |Z(R)| - |Nil(R)|$.

Now, we show that $\kappa(\mathbb{CAY}(R)) \leq |Z(R)| - |Nil(R)|$. To see this, it suffices to prove that $Z(R) \setminus Nil(R)$ is a vertex cut of $\mathbb{CAY}(R)$. Let H be the graph obtained from $\mathbb{CAY}(R)$ by removing the vertices in $Z(R) \setminus Nil(R)$. Note that the vertices of H consist of $Reg(R) \cup Nil(R)$. Now, Lemma 13 implies that no vertex of $Reg(R)$ has a neighbor in $Nil(R)$ in the graph H . Thus, $Nil(R)$ forms a non-empty (complete) connected component of graph H . Since $\mathbb{CAY}(R)$ is connected, we deduce that $Z(R) \setminus Nil(R)$ is a vertex cut. Thus, by Menger's Theorem $\kappa(\mathbb{CAY}(R)) \leq |Z(R)| - |Nil(R)|$. The assertion is proved. \square

Corollary 17. *Let R be a finite non-local ring. If R has a residue field isomorphic to \mathbb{Z}_2 , then $\kappa'(T(\Gamma(R))) = |Z(R)| - 1$ and $\kappa(T(\Gamma(R))) = |Z(R)| - |Nil(R)|$.*

Proof. Since every finite ring decomposes into a product of finite local rings (see [6, Theorem 8.7]), by Part (b) of [21, Theorem 5.2], we obtain that $T(\Gamma(R)) \cong \mathbb{CAY}(R)$. Now, the assertion follows from Theorem 16. \square

Theorem 18. *Let R be a finite non-local ring. Then $\mathbb{CAY}(R)$ is Hamiltonian.*

Proof. Since R is non-local, $|R| \geq 3$. Moreover, by Theorem 10, $\mathbb{CAY}(R)$ is connected. Thus, by [19, Corollary 3.2], $\mathbb{CAY}(R)$ is Hamiltonian. \square

3. The Quotient and the Perfectness of the Cayley Graph of a Ring

In this section, we study the quotient graph of $\mathbb{CAY}(R)$ and its relation with the quotient ring $\frac{R}{Nil(R)}$. In the end, we characterize zero-dimensional semi-local rings whose Cayley graph is perfect. First, for a graph G , we provide some background on the *quotient graph* G/S , whose properties are tightly close to those of G . Define the relation \sim on the vertices of G as follows: $x \sim y$ if and only if $N[x] = N[y]$. It is an equivalence relation on the vertices of G . Denote the equivalence class of x by $[x]$ and define a simple graph G/S with the vertex set $\{[x] : x \in G\}$ and two distinct vertices $[x]$ and $[y]$ are adjacent if and only if x and y are adjacent in G . This graph is independent of chosen representatives and it is well-defined. For more information we refer the interested reader to [14]. In the sequel, we would like to determine the equivalence class of each vertex of $\mathbb{CAY}(R)$.

Theorem 19. *Let R be a ring and $Z(R)$ be a union of finitely many minimal prime ideals of R . Then for every two elements x and y of R , the following statements are equivalent:*

- (i) $x \sim y$,
- (ii) $N[x] = N[y]$,
- (iii) $x - y \in \text{Nil}(R)$.

Proof. The equivalence of (i) and (ii) is clear from the definition. Now, we show that (ii) and (iii) are equivalent. By Lemma 13, (iii) \implies (ii). Now, suppose that $N[x] = N[y]$ and $Z(R) = \cup_{i=1}^n \mathfrak{p}_i$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \text{Min}(R)$. We claim that $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Let $\mathfrak{p} \in \text{Min}(R)$. By Lemma 3, we have $\mathfrak{p} \subseteq Z(R) = \cup_{i=1}^n \mathfrak{p}_i$. Therefore, by Prime Avoidance Theorem ([20, Theorem 3.61]), there exists $1 \leq j \leq n$ such that $\mathfrak{p} \subseteq \mathfrak{p}_j$. Since both \mathfrak{p} and \mathfrak{p}_j are minimal prime ideals, we deduce that $\mathfrak{p} = \mathfrak{p}_j$. Thus, $\text{Min}(R) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and the claim is proved. Let $A = \{i : x - y \in \mathfrak{p}_i\}$. Since $N[x] = N[y]$, $A \neq \emptyset$. If $A^c = \{1, \dots, n\} \setminus A \neq \emptyset$, by Prime Avoidance Theorem, there exists $z \in \cap_{i \in A^c} \mathfrak{p}_i$ such that $z \notin \cup_{i \in A} \mathfrak{p}_i$. Now, we show that $z - x + y$ is a regular element of R . If $z - x + y \in Z(R) = \cup_{i=1}^n \mathfrak{p}_i$, then for some $1 \leq k \leq n$, $z - x + y \in \mathfrak{p}_k$. If $k \in A$, then $z \in \mathfrak{p}_k$, a contradiction. So, $k \in A^c$. This yields that $x - y \in \mathfrak{p}_k$, a contradiction again. Hence, $z - x + y$ is a regular element of R . Thus, $z + y$ is adjacent to x in $\overline{\text{CAY}}(R)$. Now, $N[x] = N[y]$ implies $z + y$ is also adjacent to y in $\overline{\text{CAY}}(R)$. Hence, z is a regular element of R , a contradiction. Therefore, $A^c = \emptyset$ and so $x - y \in \cap_{i=1}^n \mathfrak{p}_i$. Since $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, we deduce that $x - y \in \cap_{i=1}^n \mathfrak{p}_i = \text{Nil}(R)$. The proof is complete. \square

Remark 20. Zero-dimensional semi-local rings, reduced rings with finitely many minimal prime ideals and Noetherian rings such that every associated prime ideal is a minimal prime ideal are examples of rings satisfying the assumptions of Theorem 19.

Corollary 21. *Let R be a zero-dimensional semi-local ring. Then $\text{CAY}(R)/S \cong \text{CAY}(\frac{R}{J(R)})$.*

Proof. By Theorem 19, for every $x \in R$, $[x] = x + \text{Nil}(R) = x + J(R)$. Thus, the vertex set of $\text{CAY}(R)/S$ is $\frac{R}{J(R)}$. Also, note that for every $z \in R$, by Lemma 8, $z \in Z(R)$ if and only if $z + J(R) \in Z(\frac{R}{J(R)})$ and so two distinct vertices of $\text{CAY}(R)/S$ are adjacent in $\text{CAY}(R)/S$ if and only if they are adjacent in $\text{CAY}(\frac{R}{J(R)})$. The proof is complete. \square

The *Strong Perfect Graph Theorem* states that a finite graph G is perfect if and only if neither G nor \overline{G} contains an induced odd cycle of length at least 5, see [9, Theorem 14.18]. Hence, it is easy to see that an arbitrary graph G (not necessarily finite) is perfect if and only if neither G nor \overline{G} contains an induced odd cycle of (finite) length at least 5. Thus, the Strong Perfect Graph Theorem is generalized to infinite graphs. Now, it is clear that a graph G (not necessarily finite) is perfect if and only if \overline{G} is perfect. For the finite case this is known as the *Perfect Graph Theorem* verified by Lovász, see [9, Theorem 14.12]. In the next theorem, for a zero-dimensional semi-local ring R , we study the perfectness of $\mathbb{CAY}(R)$. This theorem gives a family of infinite graphs whose every finite induced subgraph has a clique number equal to its chromatic number.

Theorem 22. *Let R be a zero-dimensional semi-local ring. Then $\mathbb{CAY}(R)$ is perfect if and only if one of the following statements holds:*

- (i) $|\text{Max}(R)| \leq 2$,
- (ii) R has a residue field isomorphic to \mathbb{Z}_2 .

Proof. We first consider the two following claims.

Claim 1. *A graph G is perfect if and only if G/S is perfect.*

To see this, suppose that G is perfect. Since every induced odd cycle of length at least 5 in G/S or $\overline{G/S}$ gives an induced odd cycle of length at least 5 in G or \overline{G} , by the Strong Perfect Graph Theorem, we conclude that G/S is perfect. Conversely, suppose that G/S is perfect. By contradiction, suppose that C is an induced odd cycle of length at least 5 in G or \overline{G} . We show that C has no two distinct vertices belonging to the same equivalence class. Let u and v be two distinct vertices of C such that $u \sim v$. Since the length of C is at least 5, there exists $x \in V(C)$ adjacent to u and not adjacent to v , a contradiction. Thus, C has no two distinct vertices belonging to the same equivalence class, i.e. the existence of an induced odd cycle of length at least 5 in G or \overline{G} implies an induced odd cycle of length at least 5 in G/S or $\overline{G/S}$, a contradiction. The proof of Claim 1 is complete.

Claim 2. *Let $n \geq 2$ be a positive integer, G_1, \dots, G_n be complete graphs (not necessarily finite) with at least two vertices and $G = G_1 \times \dots \times G_n$. If H is a finite induced subgraph in G or \overline{G} , then there exists positive integer $m_i, m_i \geq 2$, such that H is an induced subgraph either in $K_{m_1} \times \dots \times K_{m_n}$ or in $\overline{K_{m_1} \times \dots \times K_{m_n}}$, respectively.*

To see this, assume that H is a finite induced subgraph of G . Let M_i be the set of vertices of G_i that appear as the i -th component of one vertex of H . Note that if

$|M_j| = 1$, for some $1 \leq j \leq n$, then H has no edge. Since H is finite, every M_i is a finite set of vertices of G_i . So, H is an induced subgraph of $K_{|M_1|} \times \cdots \times K_{|M_n|}$. Thus, $m_i = |M_i| \geq 2$ are the desired integers. Now assume that H is an induced subgraph of \overline{G} . Hence, \overline{H} (the complement is respect to the complete graph on the vertices of G) is a finite induced subgraph of G . Therefore, by the previous argument, there exists positive integer m_i , $m_i \geq 2$, such that \overline{H} is an induced subgraph of $K_{m_1} \times \cdots \times K_{m_n}$. This implies that H is an induced subgraph of $\overline{K_{m_1} \times \cdots \times K_{m_n}}$. The proof of Claim 2 is complete.

Claim 3. *Let $n \geq 2$ be a positive integer and G_1, \dots, G_n be complete graphs (not necessarily finite) with at least two vertices. Then $G = G_1 \times \cdots \times G_n$ is perfect if and only if either $n = 2$ or $n \geq 3$ and $G_i \cong K_2$, for some i .*

The proof relies on [14, Theorem A.23] which characterizes perfectness of a finite direct product of finite graphs. Here, we are dealing with a finite direct product of possibly infinite graphs. To prove the claim, suppose that G is perfect. If one G_i has exactly two vertices, then the assertion follows. Hence, let $m_i \geq 3$ be a positive integer such that K_{m_i} is a subgraph of G_i , for $i = 1, \dots, n$. Thus, $K_{m_1} \times \cdots \times K_{m_n}$, as a subgraph of G , is a finite perfect graph. So, by [14, Theorem A.23], $n = 2$. Conversely, assume that either $n = 2$ or $n \geq 3$ and say $G_1 \cong K_2$. Let C be an induced cycle (of finite length) in G or \overline{G} . Since C is a finite graph, by Claim 2 there exists positive integer m_i , $m_i \geq 2$, such that C is an induced cycle either in $K_{m_1} \times \cdots \times K_{m_n}$ or in $\overline{K_{m_1} \times \cdots \times K_{m_n}}$, respectively. On the other hand $n = 2$ or $n \geq 3$ and $G_1 \cong K_2$. So, $n = 2$ or $n \geq 3$ and $K_{m_1} \cong K_2$. Hence by applying [14, Theorem A.23] to $K_{m_1} \times \cdots \times K_{m_n}$ and noting that finite complete graphs are also complete multipartite graphs, we conclude that $K_{m_1} \times \cdots \times K_{m_n}$ is perfect. Thus, length of C is less than 5. Hence neither G nor \overline{G} has an induced odd cycle of length at least 5. Hence, by Strong Perfect Graph Theorem, G is perfect. The proof of Claim 3 is complete.

Now, we prove the assertion. By Corollary 21 and Claim 1, it suffices to prove the assertion for $\mathbb{CAY}(\frac{R}{J(R)})$. By Corollary 4, it is equivalent to show the assertion for $G_{\frac{R}{J(R)}}$. Let $\text{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, for some $n \in \mathbb{N}$. If $n = 1$, then there is nothing to prove. Hence, assume that $n \geq 2$. Since by Chinese Remainder Theorem (see [6, Proposition 1.10]), $G_{\frac{R}{J(R)}} \cong G_{\frac{R}{\mathfrak{m}_1}} \times \cdots \times G_{\frac{R}{\mathfrak{m}_n}}$, Claim 3 implies that $G_{\frac{R}{J(R)}}$ is perfect if and only if either $n = 2$ or $n \geq 3$ and R has a residue field isomorphic to \mathbb{Z}_2 . The proof is complete. \square

By Corollary 4, the following result extends Theorem 9.5 of [3] to zero-dimensional semi-local rings.

Corollary 23. *Let R be a zero-dimensional semi-local ring. Then G_R is perfect if and only if one of the following statements holds:*

- (i) $|\text{Max}(R)| \leq 2$,
- (ii) R has a residue field isomorphic to \mathbb{Z}_2 .

4. The Induced Subgraph on the Regular Elements

Following [4], we are interested in studying $\text{Reg}(\text{CAY}(R))$. Since the multiplication by an invertible element of R is an automorphism of $\text{Reg}(\text{CAY}(R))$, by Corollary 4, we conclude that for every zero-dimensional ring R , $\text{Reg}(\text{CAY}(R))$ is a vertex transitive graph. In this section, we determine the clique number and the chromatic number of $\text{Reg}(\text{CAY}(R))$.

To study the coloring of $\text{Reg}(\text{CAY}(R))$, we need the following theorem in which we deal with rings R whose set of zero-divisors is a union of finitely many ideals of R . A ring R is said to *have few zero-divisors* if $Z(R)$ is a union of finitely many prime ideals. By [11], if R has few zero-divisors, then any overring of R i.e. any ring between R and $T(R)$, has few zero-divisors. In particular, by [20, Corollary 9.36], any overring of a Noetherian ring has few zero-divisors, which provides a large family of rings of this kind.

Theorem 24. *Let R be a ring which is not an integral domain. Suppose that $|\text{Min}(R)| < \infty$ and $Z(R)$ is a union of finitely many ideals of R . Then the following statements are equivalent:*

- (i) R is a finite ring,
- (ii) $\chi(\text{Reg}(\text{CAY}(R)))$ is finite,
- (iii) $\omega(\text{Reg}(\text{CAY}(R)))$ is finite,
- (iv) $\text{Reg}(\text{CAY}(R))$ has no infinite clique.

Proof. It is clear that (i) \implies (ii), (ii) \implies (iii) and (iii) \implies (iv). We show that (iv) \implies (i). First suppose that R is a non-reduced ring. Obviously, $1 + \text{Nil}(R) \subseteq \text{Reg}(R)$ is a clique for $\text{Reg}(\text{CAY}(R))$. Since $\text{Reg}(\text{CAY}(R))$ has no infinite clique, we deduce that $1 + \text{Nil}(R)$ is finite and so $\text{Nil}(R)$ is finite. We show that $\text{Reg}(R)$ is finite as well. Let $\{r_i\}_{i=1}^{\infty}$ be an infinite subset of $\text{Reg}(R)$ and x be a non-zero element of $\text{Nil}(R)$. Since $|\text{Nil}(R)| < \infty$, we conclude that there exists $A \subseteq \mathbb{N}$ such that A is infinite and for every $i, j \in A$, $r_i x = r_j x$. Thus, $\{r_i\}_{i \in A}$ forms an infinite clique for $\text{Reg}(\text{CAY}(R))$, a contradiction. Hence, $\text{Reg}(R)$ is finite and so by [2, Theorem 2], R is a finite ring.

Therefore, one may assume that R is a reduced ring. Let $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Thus, by Lemma 3, $Z(R) = \bigcup_{i=1}^n \mathfrak{p}_i$. We claim that for every $\emptyset \neq B \subsetneq \{1, \dots, n\}$,

$$\left| \bigcap_{i \in B} \mathfrak{p}_i \setminus \bigcup_{i \in B^c} \mathfrak{p}_i \right| < \infty.$$

By the Prime Avoidance Theorem ([20, Theorem 3.61]), $\bigcap_{i \in B^c} \mathfrak{p}_i \setminus \bigcup_{i \in B} \mathfrak{p}_i \neq \emptyset$. Let $z \in \bigcap_{i \in B^c} \mathfrak{p}_i \setminus \bigcup_{i \in B} \mathfrak{p}_i$. Now, we show that for every $x \in \bigcap_{i \in B} \mathfrak{p}_i \setminus \bigcup_{i \in B^c} \mathfrak{p}_i$, $x + z$ is a regular element of R . By contradiction, suppose that $x + z \in Z(R) = \bigcup_{i=1}^n \mathfrak{p}_i$. Hence, there exists k , $1 \leq k \leq n$, such that $x + z \in \mathfrak{p}_k$. If $k \in B$, then $z \in \mathfrak{p}_k$, a contradiction. If $k \in B^c$, then $x \in \mathfrak{p}_k$, a contradiction. Thus, $\{x + z \mid x \in \bigcap_{i \in B} \mathfrak{p}_i \setminus \bigcup_{i \in B^c} \mathfrak{p}_i\}$ forms a clique for $\text{Reg}(\mathbb{CAY}(R))$. So, the claim is proved. For B , $\emptyset \neq B \subsetneq \{1, \dots, n\}$, let $Z(B) = \bigcap_{i \in B} \mathfrak{p}_i \setminus \bigcup_{i \in B^c} \mathfrak{p}_i$. Since

$$Z(R) \setminus \bigcap_{i=1}^n \mathfrak{p}_i = \bigcup_{i=1}^n \mathfrak{p}_i \setminus \bigcap_{i=1}^n \mathfrak{p}_i = \bigcup_{\emptyset \neq B \subsetneq \{1, \dots, n\}} Z(B)$$

and $\bigcap_{i=1}^n \mathfrak{p}_i = (0)$ (see [20, Corollary 3.54]), we conclude that $Z(R)$ is finite. Since R is not an integral domain by [5, Theorem 2.2], we deduce that R is finite and the proof is complete. \square

Note that by [20, Corollary 9.36], Noetherian rings are among those families of rings satisfying the assumptions of the previous theorem. Also, by Lemma 3 and Corollary 4, reduced rings with finitely many minimal prime ideals and zero-dimensional semi-local rings are other examples of this kind of rings.

Remark 25. Let $R = \prod_{i \in \mathbb{N}} \mathbb{Z}_2$. It can be shown that R is a zero-dimensional ring with $\text{Reg}(R) = \{1\}$ and contains infinitely many minimal prime ideals.

Now, for a finite ring R , we would like to determine the clique number and the chromatic number of $\text{Reg}(\mathbb{CAY}(R))$. Before stating the results, we need the following notation. Let X and Y be two finite sets and $|X| \leq |Y|$. A *Latin rectangle of size* $|X| \times |Y|$ *over* Y , denoted by $L^{X,Y}$, is a matrix of size $|X| \times |Y|$ whose entries are in Y and entries in each row and each column are distinct. Let (R, \mathfrak{m}) be a finite local ring, $|\frac{R}{\mathfrak{m}}| = k$ and $\frac{R}{\mathfrak{m}} = \{f_1 + \mathfrak{m}, \dots, f_k + \mathfrak{m}\}$. Let x be an arbitrary element of R and $f_i + \mathfrak{m}$ be the unique element of $\{f_1 + \mathfrak{m}, \dots, f_k + \mathfrak{m}\}$ equal to $x + \mathfrak{m}$. We denote $f_i + \mathfrak{m}$ by $\pi(x)$ and $x - f_i$ by \bar{x} .

Theorem 26. *Let $R = R_1 \times \dots \times R_n$, be a finite ring, where (R_i, \mathfrak{m}_i) is a local ring. If $|\frac{R_1}{\mathfrak{m}_1}| \leq \dots \leq |\frac{R_n}{\mathfrak{m}_n}|$, then $\omega(\text{Reg}(\mathbb{CAY}(R))) = \chi(\text{Reg}(\mathbb{CAY}(R))) = |\mathfrak{m}_1|(|R_2| - |\mathfrak{m}_2|) \cdots (|R_n| - |\mathfrak{m}_n|)$.*

Proof. If (R, \mathfrak{m}) is a local ring, then by Corollary 4, $\text{Reg}(\mathbb{CAY}(R))$ is a disjoint union of $|\frac{R}{\mathfrak{m}}| - 1$ complete graphs $K_{|\mathfrak{m}|}$ and so $\chi(\text{Reg}(\mathbb{CAY}(R))) = \omega(\text{Reg}(\mathbb{CAY}(R))) = |\mathfrak{m}|$. So, suppose that $n \geq 2$. We make the two following claims:

Claim 1. $\chi(\text{Reg}(\mathbb{CAY}(R))) \leq |\mathfrak{m}_1| \cdots |\mathfrak{m}_n| \chi(\text{Reg}(\mathbb{CAY}(\frac{R_1}{\mathfrak{m}_1} \times \cdots \times \frac{R_n}{\mathfrak{m}_n})))$.

Suppose that φ is a proper vertex coloring of $\text{Reg}(\mathbb{CAY}(\frac{R_1}{\mathfrak{m}_1} \times \cdots \times \frac{R_n}{\mathfrak{m}_n}))$. We define a vertex coloring f of $\text{Reg}(\mathbb{CAY}(R))$ as follows:

$$f((x_1, \dots, x_n)) = (\overline{x_1}, \dots, \overline{x_n}, \varphi(\pi_1(x_1), \dots, \pi_n(x_n))).$$

Assume that (x_1, \dots, x_n) and (y_1, \dots, y_n) are two adjacent vertices with the same color in $\text{Reg}(\mathbb{CAY}(R))$. Thus, there exists i , $1 \leq i \leq n$, such that $x_i - y_i \in \mathfrak{m}_i$. Hence, $\pi_i(x_i) = \pi_i(y_i)$. Since $\varphi((\pi_1(x_1), \dots, \pi_n(x_n))) = \varphi((\pi_1(y_1), \dots, \pi_n(y_n)))$, we deduce that $\pi_j(x_j) = \pi_j(y_j)$, for every j , $1 \leq j \leq n$. This together with $f((x_1, \dots, x_n)) = f((y_1, \dots, y_n))$ implies that $(x_1, \dots, x_n) = (y_1, \dots, y_n)$. Thus, f is a proper vertex coloring of $\text{Reg}(\mathbb{CAY}(R))$ and the claim is proved.

Claim 2. $\chi(\text{Reg}(\mathbb{CAY}(F_1 \times \cdots \times F_n))) \leq |F_2^*| \cdots |F_n^*|$, where $n \geq 2$ and F_i is a finite field with $|F_1| \leq \cdots \leq |F_n|$.

Let $L_1^{F_1^*, F_i^*}$ be a Latin rectangle of size $|F_1^*| \times |F_i^*|$ over F_i^* , for $2 \leq i \leq n$. We define a vertex coloring g on $V(\text{Reg}(\mathbb{CAY}(F_1 \times \cdots \times F_n))) = F_1^* \times \cdots \times F_n^*$ as follows:

$$g(x_1, \dots, x_n) = (L_{x_1 x_2}^{F_1^*, F_2^*}, \dots, L_{x_1 x_n}^{F_1^*, F_n^*}),$$

where $L_{x_1 x_i}^{F_1^*, F_i^*}$ denotes the (x_1, x_i) -entry of $L_1^{F_1^*, F_i^*}$. Now, suppose that (x_1, \dots, x_n) and (y_1, \dots, y_n) are two distinct adjacent vertices with the same color in $\text{Reg}(\mathbb{CAY}(F_1 \times \cdots \times F_n))$. Hence, there exists t , $1 \leq t \leq n$, such that $x_t = y_t$. Since $L_{x_1 x_i}^{F_1^*, F_i^*} = L_{y_1 y_i}^{F_1^*, F_i^*}$, for every i , $2 \leq i \leq n$, we deduce that $x_1 = y_1$. Therefore, $L_{x_1 x_i}^{F_1^*, F_i^*} = L_{y_1 y_i}^{F_1^*, F_i^*}$ for every i , $2 \leq i \leq n$. This implies that $(x_1, \dots, x_n) = (y_1, \dots, y_n)$, a contradiction. So, g is a proper vertex coloring of $\text{Reg}(\mathbb{CAY}(F_1 \times \cdots \times F_n))$ and the claim is proved.

Since $|\frac{R_1}{\mathfrak{m}_1}| \leq \cdots \leq |\frac{R_n}{\mathfrak{m}_n}|$, it follows by these two claims that $\chi(\text{Reg}(\mathbb{CAY}(R))) \leq |\mathfrak{m}_1|(|R_2| - |\mathfrak{m}_2|) \cdots (|R_n| - |\mathfrak{m}_n|)$. Since $(1 + \mathfrak{m}_1) \times (R_2 \setminus \mathfrak{m}_2) \times \cdots \times (R_n \setminus \mathfrak{m}_n)$ forms a clique for $\text{Reg}(\mathbb{CAY}(R))$, we conclude that $\omega(\text{Reg}(\mathbb{CAY}(R))) \geq |\mathfrak{m}_1|(|R_2| - |\mathfrak{m}_2|) \cdots (|R_n| - |\mathfrak{m}_n|)$ and so the assertion is proved. \square

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